Preprint INRNE-TH-93/4 (May 1993)
e-print quant-ph/0001028

GENERALIZED INTELLIGENT STATES AND $SU(1,1)$ AND $SU(2)$ SQUEEZING*

D.A. Trifonov
Institute for Nuclear Research and Nuclear Energy
Blv. Tzarigradsko chaussee 72, 1784 Sofia, Bulgaria

February 22, 2000

Abstract

A sufficient condition for a state $|\psi\rangle$ to minimize the Robertson-Schrödinger uncertainty relation for two observables A and B is obtained which for A with no discrete spectrum is also a necessary one. Such states, called generalized intelligent states (GIS), exhibit arbitrarily strong squeezing (after Eberly) of A and B. Systems of GIS for the $SU(1,1)$ and $SU(2)$ groups are constructed and discussed. It is shown that $SU(1,1)$ GIS contain all the Perelomov coherent states (CS) and the Barut and Girardello CS while the Bloch CS are subset of $SU(2)$ GIS.

PACS' numbers 03.65.Ca; 03.65.Fd; 42.50.Dv .

1 Introduction

The squeezed states of electromagnetic field in which the fluctuations in one of the quadrature components Q and P of the photon annihilation operator $a = (Q+iP)/\sqrt{2}$ are smaller than those in the ground state $|0\rangle$ have atracted due attention in the last decade (see for example the review papers[1, 2] and references there in). In the recent years an interest is devoted to the squeezed states for other observables[3]–[11]. One looks for non gaussian states which exhibit Q-P squeezing[3]–[7] and/or for states in which the fluctuations of other physical observables are squeezed[7]–[11].

The aim of the present paper is to construct $SU(1,1)$ and $SU(2)$ squeezed intelligent states and to consider some general properties of squeezing for an arbitrary pair of quantum observables A and B in states which minimize the Robertson-Schrödinger uncertainty

*This preprint was sent [with the here preserved mis-spellings Heizenberg, studed, ..., and the false degeneracy of the eigenvalue of $L(\lambda)$] to Phys. Rev. Lett. in May 1993 (LF5064/ 03 Jun 93) and declined from PRL in August 1993. An extended version of it appeared later in J. Math. Phys. **35**, 2297 (1994). Meanwhile similar (but not all) results were published by other authors in PRL and Phys. Rev. A.

relation (R-S UR)[12]. We call such states generalized intelligent states (GIS) or squeezed intelligent states when the accent is on their squeezing properties. The Q-P GIS are well studed and known as squeezed states, two photon coherent states (CS) (see references in[1, 2]), correlated states[13] or Schrödinger minimum uncertainty states[14]. The term intelligent states (IS)[11] is refered to states that provide the equality in the Heizenberg UR for A and B. The Q-P IS are also known as Heizenberg minimum uncertainty states. The spin IS are introduced and studed in[11].

2 Generalized intelligent states

For any two quantum observables A and B the corresponding second momenta in a given state obey the R-S UR[12, 13],

$$\sigma_A^2\,\sigma_B^2 \geq \frac{1}{4}(\langle C\rangle^2 + 4\sigma_{AB}^2), \quad C \equiv -i[A,B], \tag{1}$$

where σ_A, σ_B and σ_{AB} are the dispersions and the covariation of A and B,

$$\begin{aligned} \sigma_A^2 &= \langle A^2\rangle - \langle A\rangle^2, \\ \sigma_{AB} &= \frac{1}{2}(\langle AB + BA\rangle) - \langle A\rangle\langle B\rangle. \end{aligned} \tag{2}$$

The states that provide the equality in the R-S UR (1) will be called here generalized intelligent states (GIS). When the covariation $\sigma_{AB} = 0$ then the S-R UR coincides with the Heizenberg one. In paper[13] it was proved that if a pure state $|\psi\rangle$ with nonvanishing dispersion of the operator A minimizes the R-S UR then it is an eigenstate of the operator $\lambda A + iB$, where λ is a complex number, related to $\langle C\rangle$ and to $\sigma_i(\psi), i = A, B, AB$. Here we prove that this is a sufficient condition for any state $|\psi\rangle$.

Proposition 1 *A state $|\psi\rangle$ minimizes the R-S UR (1) if it is an eigenstate of the operator $L(\lambda) = \lambda A + iB$,*

$$L(\lambda)|z,\lambda\rangle = z|z,\lambda\rangle, \tag{3}$$

where the eigenvalue z is a complex number.

Proof. Let first restrict the parameter λ in the eigenvalue eqn. (3), $\mathrm{Re}\,\lambda \neq 0$. Then we express A and B in terms of $L(\lambda)$ and $L^\dagger(\lambda)$ and obtain

$$\begin{aligned} \sigma_A^2(z,\lambda) = \frac{\langle C\rangle}{2\mathrm{Re}\,\lambda}, \qquad \sigma_B^2(z,\lambda) = |\lambda|^2\frac{\langle C\rangle}{2\mathrm{Re}\,\lambda}, \\ \sigma_{AB}(z,\lambda) = -\langle C\rangle\frac{\mathrm{Im}\,\lambda}{2\mathrm{Re}\,\lambda}, \end{aligned} \tag{4}$$

where $\langle C\rangle = \langle\lambda,z|C|z,\lambda\rangle$. The obtained second momenta (4) obey the equality in R-S UR (1).

Let now the eigenvalue equation (3) holds for $\mathrm{Re}\,\lambda = 0$. This means that the state $|z,\lambda\rangle$ is an eigenstate of the Hermitean operator $rA + B$ where $r = \mathrm{Im}\,\lambda$. We consider now the mean value of the non negative operator $F^\dagger(r)F(r)$, where $F(r) = rA+B-(r\langle A\rangle+\langle B\rangle)$ and r is any real number. Herefrom we get the uncertainty relation

$$\sigma_A^2\,\sigma_B^2 \geq \sigma_{AB}^2\,, \tag{5}$$

the equality holding in the eigenstates of $F(r)$ only. One can consider the equality in (5) as the desired equality in the Robertson-Schrödinger UR if in these states the mean value of the operator C vanishes. And this is the case. Indeed, consider in $|z, ir\rangle$ the mean values of the operators $A(rA+B)$ and $(rA+B)A$. We easily get the coinsidence of the two mean values, wherefrom we obtain $\langle ir, z|C|z, ir\rangle = 0$.

Thus all eigenstates $|z,\lambda\rangle$ are GIS. One can prove that when the operator A has no discrete spectrum then for any $|\psi\rangle$ $\sigma_A(\psi) \neq 0$, thereby the condition (3) is also necessary and all A-B GIS (for any B) are of the form $|z,\lambda\rangle$. Such are for example the cases of canonical Q-P GIS[14] and the $SU(1,1)$ GIS, considered below. The above result stems from the following property of the dispersion of quantum observables:

$$\sigma_A(\psi) = 0 \Longleftrightarrow A|\psi\rangle = a|\psi\rangle. \tag{6}$$

As a consequence of the second part of the proof of the Proposition 1 we have the following

Proposition 2 *If the commutator $C = -i[A,B]$ is a positive operator then the operator $rA+B$ with real r has no eigenstates in the Hilbert space.*

In terms of GIS $|z,\lambda\rangle$ the above Proposition 2 gives the restriction on λ: $\mathrm{Re}\,\lambda \neq 0$ in cases of positive C.

Before going to examples let us point out that the A-B IS $|z, \lambda = 1\rangle \equiv |z\rangle$ are noncorrelated and with equal variances,

$$L|z\rangle = z|z\rangle, \qquad L = L(\lambda=1) = A+iB, \tag{7}$$

$$\sigma_A^2(z) = \frac{1}{2}\langle z|C|z\rangle = \sigma_B^2(z). \tag{8}$$

We shall call such states equal variances IS or non squeezed IS, addopting the Eberly and Wodkiewicz[7] definition of A-B squeezed states. It is convenient to describe this squeezing by means of the dimensionless parameter q_A[8]

$$q_A = \frac{\langle C\rangle/2\,-\,\sigma_A^2}{\langle C\rangle/2}, \tag{9}$$

in terms of which the 100% squeezing corresponds to $q_A = 1$. In the equal variances IS $|z\rangle$ $q_A = 0 = q_B$.

Let now consider the cases when the commutator $C = -i[A,B]$ is a positive operator: $\langle\psi|C|\psi\rangle > 0$. In such cases $\mathrm{Re}\lambda \neq 0$ and we can safely devide by $\langle\psi|C|\psi\rangle$. Then from eqns (4) we get the quite general result for squeezing in GIS $|z,\lambda\rangle$ with positive C,

$$q_A(z,\lambda) = 1 - \frac{1}{2\mathrm{Re}\,\lambda}, \qquad q_B(z,\lambda) = 1 - \frac{|\lambda|^2}{2\mathrm{Re}\,\lambda}. \tag{10}$$

We see that the squeezing parameter q depends on λ only and 100% squeezing of A is obtained at $\mathrm{Re}\,\lambda \to \infty$ (and of B at $\lambda = 0$).

In many cases the IS $|z\rangle$ are constructed. Except of the canonical Q-P case we point out also the cases of lowering and raising operators of some semisimple Lie groups (the $SU(2)$ and the $SU(1,1)$[15] for example) and for the quantum group $SU(1,1)_q$, constructed recently[10]. The GIS $|z,\lambda\rangle$ are eigenstates of the linearly transformed operator

$$L \longrightarrow L(\lambda) = uL + vL^\dagger, \tag{11}$$

where $u = (\lambda+1)/2$, $v = (\lambda-1)/2$, $L^\dagger = A - iB$. If this is a similarity transformation then GIS can be obtained by acting on $|z\rangle$ with the transforming operator $S(\lambda)$ (the generalized squeezing operator) as it was done by Stoler (see the reference in[1, 2]) in the canonical case. In the examples below we construct GIS by solving the eigenvalue equations of $L(\lambda)$.

3 $SU(1,1)$ squeezed intelligent states

In this section we construct and discuss K_1-K_2 GIS, where K_1 and K_2 are the generators of the discrete series $D^+(k)$ of representations of $SU(1,1)$ with Cazimir operator $C_2 := k(k-1)$. From the commutation relation $[K_1, K_2] = -iK_3$ we see that one can apply the corresponding formulas of the previous section with $A = K_1, B = -K_2$ and $C = K_3$. The operator K_3 is positive with eigenvalues $k+m$ where $m = 0,1,2,\ldots,$. Then as a consequence of the Proposition 2 the GIS $|z,\lambda;k\rangle$ exist only if $\mathrm{Re}\,\lambda \neq 0$ and one can safely use formulas (4) for the second momenta of $K_{1,2}$ in the $SU(1,1)$ GIS $|z,\lambda;k\rangle$. Since the operator K_1 has no discrete spectrum the condition (3) is also necessary for GIS.

The $SU(1,1)$ equal variances IS $|z;k\rangle$ (the eigenstates of $K_1 - iK_2 \equiv K_-$) have been constructed and studed by Barut and Girardello as 'new "coherent" states associated with noncompact groups'[15]. These states form an overcomplete family of states and provide a representation of any state $|\psi\rangle$ in terms of entire annalytic function $\langle\psi|z;k\rangle$ of z of order 1 and type 1 (exponential type). In the Hilbert space of such entire analytic functions the generators of $SU(1,1)$ act as the following differential operators [15] (we shall call this BG-representation)

$$\begin{aligned} K_3 = k + z\frac{d}{dz}\,, \quad K_+ = K_-^\dagger = z\,, \\ K_- = 2k\frac{d}{dz} + z\frac{d^2}{dz^2}\,. \end{aligned} \tag{12}$$

We use the BG-representation to construct the $SU(1,1)$ GIS $|z',\lambda;k\rangle$ (we denote for a while the eigenvalue by z'). The eigenvalue equation (3) now reads

$$\left[u(2k\frac{d}{dz} + z\frac{d^2}{dz^2}) + vz\right]\Phi_{z'}(z) = z'\Phi_{z'}(z)\,, \tag{13}$$

where the parameters u, v have been defined in formula (11). By means of a simple substitutions the above equation is reduced to the Kummer equation for the confluent hypergeometric function ${}_1F_1(a,b;z)$ [16], so that we have the following solution of eqn. (13)

$$\Phi_{z'}(z) = \exp{(cz)}\,{}_1F_1(a,b;-2cz)\,, \tag{14}$$

$$a = k - \frac{z'}{2uc}\,, \quad b = 2k; \quad c^2 = -\frac{v}{u}\,. \tag{15}$$

This solution obey the requirements of the BG representation iff

$$|c| = \sqrt{|v/u|} < 1 \Leftrightarrow \mathrm{Re}\,\lambda > 0\,, \tag{16}$$

which is exactly the restriction on λ imposed by the positivity of the commutator $C \equiv K_3$, according to the Proposition 2. No other constrains on z' and λ are needed. Thus we obtain the $SU(1,1)$ GIS $|z',\lambda;k\rangle$ in the BG-representation in the form

$$\langle k;\lambda,z'|z;k\rangle = \exp{(c^*z)}\,{}_1F_1(a^*,b;-2c^*z)\,, \tag{17}$$

where the parameters a, b and c are given by formulas (3.4). Using the power series of ${}_1F_1(a,b;z)$[16] we get the coinsidence of our solution (17) at $\lambda = 1$ ($u = 1$, $v = 0$) with the solution of Barut and Girardello[15],

$$\langle k;\lambda = 1,z'|z;k\rangle = {}_0F_1(2k;zz'^*) \;=\; \langle k;z'|z;k\rangle. \tag{18}$$

We note the twofold degeneracy of the eigenvalues of the operator $L(\lambda \neq 1)$ as it is seen from eqn. (3.4). We denote the two solutions as $\langle \pm;k;\lambda,z'|z;k\rangle$. The degeneracy is removed at $\lambda = 1$ as it is known from the BG-solution. Thus this point is a branching point for the operator $L(\lambda)$. It worth noting that the degeneracy is also removed by the following constrain on the two complex parameters z' and λ in eqn. (3.6)

$$z' \;=\; 2k\sqrt{-uv} \;=\; k\sqrt{1-\lambda^2}\,. \tag{19}$$

Using the properties of the function ${}_1F_1(a,b;z)$ [16] we get from (17) in both ($\pm$) cases the same expression $\exp{(z\sqrt{-v^*/u^*})}$ which can be seen to be nothing but the BG-representation of the Perelomov $SU(1,1)$ CS $|\zeta;k\rangle$[17] with $\zeta = \sqrt{-v/u}$,

$$|\zeta;k\rangle = (1-|\zeta|^2)^k\, \exp{(\zeta K_+)}\,|k;k\rangle\,. \tag{20}$$

If we impose $z' = -2k\sqrt{-uv}$ we get CS$|-\zeta;k\rangle$. One can directly check (using the $SU(1,1)$ commutation relations only) that CS (20) are indeed eigenstates of $L(\lambda)$, eqn. (11), with eigenvalue (19) provided $\zeta^2 = -v/u$. We calculate explicitly the first and second momenta of the generators K_i in CS $|\zeta;k\rangle$ (for σ_{K_i} see also[8])

$$\begin{gathered}\sigma_{K_1K_2} = -2k\,\frac{\mathrm{Re}\,\zeta\,\mathrm{Im}\,\zeta}{(1-|\zeta|^2)^2}\,,\\ \sigma^2_{K_1} = \frac{k}{2}\,\frac{|1+\zeta^2|^2}{(1-|\zeta|^2)^2}, \quad \sigma^2_{K_2} = \frac{k}{2}\,\frac{|1-\zeta^2|^2}{(1-|\zeta|^2)^2}\end{gathered} \tag{21}$$

and convince that the equality in the R-S UR (1) is satisfied.

Thus all the Perelomov $SU(1,1)$ CS are GIS. They are represented by the points of the two dimensional surface (19) in the four dimensional space of points (z,λ). The BG CS[15] form another subset of $SU(1,1)$ GIS isomorfic to the plane $\lambda = 1$.

We note that the aboved formulas for the first and second momenta of K_i in CS $|\zeta;k\rangle$ hold also for the (non square integrable) Lipkin-Cohen representation with Bargman index $k = 1/4$ (but not for $k = 3/4$),

$$K_1 \,=\, \frac{1}{4}\,(Q^2 - P^2), \quad K_2 \,=\, -\frac{1}{4}(QP + PQ),$$
$$K_3 \,=\, \frac{1}{4}(Q^2 + P^2). \tag{22}$$

Due to the expressions of K_i in terms of the canonical pair Q, P the CS $|\zeta; k = 1/2, 1/4, 3/4\rangle$ ($|\zeta; k = 1/4, 3/4\rangle$ are eigenstates of the squared boson operator a^2) are of interest for Q-P squeezing[4, 14, 18]. One can also calculate the fluctuations of Q and P[18] and show that CS $|\zeta; k = 1/4\rangle$ exhibit about 56% ordinary squeezing (Bužek[4]). The squeezing of $K_{1,2}$ in CS $|\zeta;k\rangle$ has been studed in[8]: the 100% squeezing (in the sense of the parameter q, eqn. (9) for K_1 is obtained at $\zeta = i$. We note however that

$$\sigma_i^2(\zeta;k) \,\geq\, \frac{k}{2} \,=\, \sigma_i^2(0;k), \quad i = K_1, K_2\,,$$

i.e. no squeezing of σ_i in $|\zeta;k\rangle$ in comparison with the ground state $|0;k\rangle$.

In conclusion to this section we note that for $SU(1,1)$ GIS the squeezing operator $S(\lambda)$ exists and can be defined by means of the relation $|z,\lambda;k\rangle = S(\lambda)|z;k\rangle$ since the spectra of L and $L(\lambda)$ coinside. It belongs again to the $SU(1,1)$ (but not to the series $D^+(k)$ since one can show that it is not unitary) and its matrix elements $\langle k;z|S|z;k\rangle$ are explicitly given by the functions (17) with $z' = z$. These diagonal matrix elements determine S uniquely due to the analyticity property of the BG-representation[15]. We recall that the same property of the diagonal matrix elements holds in the canonical (Glauber) CS representation (see for example[2] and references therein).

4 $SU(2)$ squeezed intelligent states

Let now A, B and C be the generators $J_1, -J_2$ and $-J_3$ of $SU(2)$ group, i.e. the spin operators of spin $j = 1/2, 1, \ldots,$. In this example the commutator $C = -J_3$ is not positive (the limit $\mathrm{Re}\,\lambda = 0$ can be taken) and the operator $A = J_1$ has a disctete spectrum (some of its eigenstates are examples of exceptional GIS which are not eigenstates of $L(\lambda)$). In paper[11] there were constructed the eigenstates (in their notations) $|w_N(\tau)\rangle$ of the operator $J(\alpha) = J_1 - i\alpha J_2$, where $N = 0, 1, 2\ldots, 2j$, $\tau^2 = (1-\alpha)/(1+\alpha)$, α being arbirary complex number. These states are eigenstates also of $L(\lambda) = \lambda J_1 - iJ_2$, thereby they all are J_1-J_2 GIS, minimizing the R-S UR (1). They can be represented in the general form $|z_N,\lambda;j\rangle$ with the eigenvalues $z_N = (j-N)\sqrt{\lambda^2-1}$. Among them (for $N = 0$ and $N = 2j$) are the Bloch (the spin or the $SU(2)$) CS $|\tau;-j\rangle$ and $|-\tau;-j\rangle$ (τ is any complex number)

$$|\tau;-j\rangle = (1+|\tau|^2)^{-j}\exp{(\tau J_+)}|-j\rangle. \tag{23}$$

The mean values of $J_i, i = 1,2,3$ and J_i^2 (and the dispersions σ_{J_1} and σ_{J_2}) in Bloch CS are known[11, 19]. Calculating also the covariation,

$$\sigma_{J_1,J_2}(\tau) = 2j\frac{\mathrm{Re}\,\tau\,\mathrm{Im}\,\tau}{(1+|\tau|^2)^2} \tag{24}$$

we can directly check that in CS $|\tau\rangle$ the equality in the R-S UR (1) holds for the spin operators $J_{1,2}$. Thus the Bloch CS are a subset of the $SU(2)$ GIS.

Let us briefly discuss the properties of the $SU(2)$ GIS. First of all for a given parameter λ there are $2j+1$ independent GIS $|z_N, \lambda; j\rangle$. There is only one equal variances IS, namely $|-j\rangle$, the point $\lambda = 1$ being again the branching point of the $L(\lambda)$. From this fact it follows that squeezing operator does not exist. Since the commutator $C = i[J_1, J_2] = -J_3$ the limit $\mathrm{Re}\,\lambda = 0$ in GIS is alowed and in the fluctuations formulas (4) as well since at this limit $\langle C\rangle = \langle J_3\rangle = 0$. The operator $A = J_1$ has a discrete spectrum, therefore $\sigma_A \geq 0$. From the explicit formula

$$\sigma^2_{J_1}(\tau) = \frac{j}{2}\frac{|1-\tau^2|^2}{(1+|\tau|^2)^2} \tag{25}$$

we see that this fluctuation vanishes at $\tau^2 = 1$. Therefore in virture of the property (6) the Bloch CS $|\tau = \pm 1; -j\rangle$ are eigenstates of J_1 which can be checked also directly, the eigenvalues being $\pm j$. The other eigenstates of J_1 are exactly those exceptional states which minimize the R-S UR (1) but are not of the form $|z, \lambda\rangle$ (i.e. dont obey eqn.(3)). The final note we make about $SU(2)$ GIS is that except for the eigenvalue $z_N = 0$ (when $N = j$) all the others are not degenerate (unlike the $SU(1,1)$ case).

5 Concluding remarks

We have presented a method for construction of squeezed intelligent states (called here generalized intelligent states (GIS)) for any two quantum observables A and B in which 100% squeezing (after Eberly) can be obtained. GIS minimize the Robertson-Schrödinger uncertainty relation and can be considered as a generalization of the canonical Q-P squeezd states[13]. When the operators A and/or B are exspressed in terms of the canonical pair Q, P one can look in the A-B GIS for the squeezing of Q end/or P as well. Such are for example the cases of $SU(1,1)$ GIS for the representations with Bargman indexes $k = 1/4, 1/2, 3/4$. The $SU(1,1)$ GIS form a larger set of states which contains as two different subsets the Perelomov CS and the Barut and Girrardello CS.

The method is based on the minimization of the Robertson-Schrödinger UR (1) for which the eigenvalue equation (3) for the operator $L(\lambda) = \lambda A + iB$ is a sufficient condition. In case of A with continuous spectrum this is also a necessary conditon independently on B. In view of this the method provides the possibility (when one is interested in squeezing of the fluctuations of A) to look for the best squeezing partner of A. Thus for example if $A = P$ then one can show that the eigenstates of $L(\lambda)$ exist for a series $B = Q^n$, $n = 1, 5, 9, \ldots,$.

When the A-B GIS can be obtained from the equal variances IS $|z\rangle$ by means of the invertable squeezing operator $S(\lambda)$ the latter belongs to $SU(1,1)$ as it can be derived from (11). This fact shows that $SU(1,1)$ plays important role in a wide class of squeezing phenomina (not only in Q-P case).

Acknowledgments

This work is partialy supported by Bulgarian Science Foundation research grant # F-116.

References

[1] R. Loudon and P. Knight, J. Mod. Opt. **34**, 709 (1987).

[2] W. Zhang, D. Feng and R. Gilmore. Rev. Mod. Phys. **62**, 867 (1990).

[3] G. D'Ariano, M. Rasetti and M. Vadacchino, Phys. Rev. D **32**, 1034 (1985).

[4] V. Bužek. J. Mod. Opt. **37**, 159 (1990); J. Sun et all. Phys. Rev. A**44**, 3369 (1991); C. Gerry and E. Hach III, Phys. Lett. A**174**, 185 (1993).

[5] J. Katriel et all, Phys. Rev. D**34**, 2332 (1986).

[6] P. Kral, J. Mod. Opt. **37**, 889 (1990).

[7] K. Wodkiewicz and J. Eberly, J. Opt. Soc. Am. B**2**, 458 (1985); K. Wodkiewicz, J. Mod. Opt. **34**, 941 (1987).

[8] V. Bužek, J. Mod. Opt. **37**, 303 (1990).

[9] J. Vaccaro and D. Pegg, J. Mod. Opt. **37**, 17 (1990).

[10] L. Kuang and F. Wang, Phys. Lett. A**173**, 221 (1993).

[11] C. Aragone, E Chalband and S. Salamo, J. Math. Phys. **17**, 1963 (1976).

[12] H. Robertson, Phys. Rev. **35**, 667 (1930); S. Schrödinger, Ber. Kil. Acad. Wiss., s. 296, Berlin (1930).

[13] V. Dodonov, E. Kurmyshev and V. Man'ko, Phys. Lett. A**76**, 150 (1980).

[14] D. A. Trifonov, J. Math. Phys. **34**, 100 (1993).

[15] A. O. Barut and L. Girardello, Commun. Math. Phys. **21**, 41 (1971).

[16] *Handbook on Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (National Bureau of Standarts, 1964; Russian translation, Nauka, 1979).

[17] A. M Perelomov, Commun. Math. Phys. **26**, 222 (1972).

[18] B. A. Nikolov and D. A. Trifonov, Commun. JINR. E2-81-798 (Dubna, 1981).

[19] E. H. Lieb, Commun. Math. Phys. **31**, 327 (1973).